Available online at www.sciencedirect.com
SCIENCE DIRECT
An International Journal
computers & mathematics with applications
Computers and Mathematics with Applications 52 (2006) 1623–1626
www.elsevier.com/locate/camwa

Dynamic Oligopolies with Intertemporal Demand Interaction

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(Received June 2005; revised and accepted October 2006)

Abstract—Single-product oligopolies without product differentiation are examined under the additional assumption that there is inter temporal demand interaction over time in the market.Therefore the market price depends on the current total production of the industry and also on a cumulated effect of earlier demands of the market. The associated dynamic model will first be derived and then the asymptotical behavior of the equilibrium will be examined.

Keywords—Oligopolies, Stability, Demand interaction, Equilibrium.

1. INTRODUCTION

Oligopoly models are the most frequently examined subjects in the literature of mathematical economics. Based upon the pioneering work of Cournot [1], many researchers have developed the different variants of oligopoly models. A comprehensive summary of single-product models and related results are given in [2], and their multi-product extensions are presented in [3]. In both static and dynamic oligopolies, the inverse demand function relates the demand and price in the same time period. However in many instances the demand for a durable goods in one period will have an effect on the demand and price of the goods in later time periods. Even in the case of non-durable goods, the demand for and consumption of a good in earlier periods will lead to taste or habit formation of consumers that will affect their later demand. In the analysis of international trade, many researchers have considered inter temporal demand interaction (see for example, [4,5]). More recently, Okuguchi and Szidarovszky [6] have developed a two-stage oligopoly and examined the existence and uniqueness of the Nash equilibrium.

In this paper, we will first introduce the dynamic extension of the model of [6], and then the asymptotical behavior of the equilibrium will be analyzed. We will derive local asymptotical stability conditions.

2. THE MATHEMATICAL MODEL

Consider an n-firm single-product oligopoly without product differentiation. Let x_k be the output of firm k, and $C_k(x_k)$ its cost. Market saturation, habit formation, etc., of earlier time periods are condensed into variable Q, which is assumed to follow a dynamic rule

0898-1221/06/$ - see front matter Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX
doi:10.1016/j.camwa.2006.10.019

$$\dot{Q} = H\left(\sum_{k=1}^{n} x_k, Q\right), \tag{1}$$

where H is a given function. The market price is also assumed to depend on both the total production of the industry and Q. Therefore, the profit of firm k can be written as

$$\Pi_k = x_k f\left(x_k + S_k, Q\right) - C_k\left(x_k\right), \tag{2}$$

where $S_k = \sum_{l \neq k} x_l$ and f is the price function. Assume that the best reply of firm k is interior, and functions f and C_k are twice continuously differentiable. Then at the optimum

$$x_k f'\left(x_k + S_k, Q\right) + f\left(x_k + S_k, Q\right) - C_k'\left(x_k\right) = 0. \tag{3}$$

Assume that

(A) $f_S' - C_k'' < 0$;
(B) $f_S' + x_k f_{SS}'' \leq 0$ and $f_Q' + x_k f_{SQ}'' \leq 0$.

Then Π_k is strictly concave in x_k. Assume that optimum exists for all S_k, $Q \geq 0$. It is also clear that the left hand side of (3) is strictly decreasing in x_k, so x_k is a unique function of S_k and Q: $x_k = R_k(S_k, Q)$. By implicit differentiation we have

$$r_k = \frac{\partial R_k}{\partial S_k} = -\frac{f_S' + x_k f_{SS}''}{2f_S' + x_k f_{SS}'' - C_k''} \in (-1, 0] \qquad \text{and} \qquad \bar{r}_k = \frac{\partial R_k}{\partial Q} = -\frac{f_Q' + x_k f_{SQ}''}{2f_S' + x_k f_{SS}'' - C_k''} \leq 0.$$

We also assume that

(C) $h = \frac{\partial H}{\partial S} > 0$ and $\bar{h} = \frac{\partial H}{\partial Q} \leq 0$.

As it is usual in the theory of dynamic oligopolies, we assume that each firm adjusts its output in the direction forward its best reply. Then the output of firm k is driven by the differential equation

$$\dot{x}_k = K_k\left(R_k\left(\sum_{l \neq k} x_l, Q\right) - x_k\right), \tag{4}$$

where $K_k > 0$ is a constant which is called the speed of adjustment of firm k. Equations (1) and (4) define a continuous system with state variables $x_1, x_2, \ldots, x_n$, Q. Clearly $x_1^*, x_2^*, \ldots, x_n^*$, Q^* is a positive equilibrium of the system if and only if

$$H\left(\sum_{k=1}^{n} x_k^*, Q^*\right) = 0 \tag{5}$$

and

$$x_k^* = R_k\left(\sum_{l \neq k} x_l^*, Q^*\right). \tag{6}$$

The asymptotical properties of the equilibrium of this dynamic system will be examined in the next section.

3. STABILITY ANALYSIS

System (1), (4) is nonlinear, the local asymptotical behavior of which is examined by linearization. The Jacobian of the system has the special form

$$\begin{pmatrix} -K_1 & K_1 r_1 & \cdots & K_1 r_1 & K_1 \bar{r}_1 \\ K_2 r_2 & -K_2 & \cdots & K_2 r_2 & K_2 \bar{r}_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_n r_n & K_n r_n & \cdots & -K_n & K_n \bar{r}_n \\ h & h & \cdots & h & \bar{h} \end{pmatrix}$$

with eigenvalue equations

$$-K_k u_k + K_k r_k \sum_{l \neq k} u_l + K_k \bar{r}_k v = \lambda u_k \qquad (l \leq k \leq n) \tag{7}$$

$$h\sum_{k=1}^{n} u_k + \bar{h}v = \lambda v. \tag{8}$$

By introducing the notation $U = \sum_{k=1}^{n} u_k$, these equations can be rewritten as

$$K_k r_k U = (\lambda + K_k + K_k r_k)\, u_k - K_k \bar{r}_k v \tag{9}$$

and

$$U = \frac{\lambda - \bar{h}}{h} v. \tag{10}$$

From these relations we have

$$u_k = \frac{K_k r_k \left(\lambda - \bar{h}\right) + K_k \bar{r}_k h}{h\left(\lambda + K_k\left(1 + r_k\right)\right)} v, \tag{11}$$

where we assume that $\lambda \neq -K_k(1+r_k)$. Since a negative eigenvalue cannot destroy asymptotical stability, this assumption does not affect the stability analysis. By adding (11) for all values of k and using (10), we obtain a single equation for v:

$$\left(\sum_{k=1}^{n} \frac{K_k r_k \left(\lambda - \bar{h}\right) + K_k \bar{r}_k h}{h\left(\lambda + K_k\left(1 + r_k\right)\right)} - \frac{\lambda - \bar{h}}{h}\right) v = 0.$$

We may assume that $v \neq 0$, as otherwise from (11), all u_k would be zero, and eigenvectors must be nonzero. Therefore, the roots of equation

$$\sum_{k=1}^{n} \frac{K_k r_k \lambda + K_k \left(\bar{r}_k h - r_k \bar{h}\right)}{\lambda + K_k\left(1 + r_k\right)} = \lambda - \bar{h} \tag{12}$$

determine if the equilibrium is locally asymptotically stable or not. Our main result can be formulated as follows.

THEOREM 3.1. *Assume that Conditions (A), (B), (C) are satisfied, and at least one of the quantities $\bar{r}_k$, $k = 1, 2, \dots, n$, and $\bar{h}$ is nonzero. Then all roots of equation (12) have negative real parts implying the local asymptotic stability of the equilibrium.*

PROOF. Assume that $\lambda = A + iB$ is a root of equation (12), with $A \geq 0$. If $p(\lambda)$ and $q(\lambda)$ denote the left- and right-hand sides of equation (12), then

$$\operatorname{Re} q(\lambda) = A - \bar{h} \geq 0 \qquad \text{and}$$

$$\begin{aligned}
\operatorname{Re} p(\lambda) &= \operatorname{Re} \sum_{k=1}^{n} \frac{K_k r_k A + K_k\left(\bar{r}_k h - r_k \bar{h}\right) + K_k r_k B i}{A + K_k(1 + r_k) + iB} \\
&= \sum_{k=1}^{n} \frac{\left[K_k r_k A + K_k\left(\bar{r}_k h - r_k \bar{h}\right)\right]\left[A + K_k\left(1 + r_k\right)\right] + K_k r_k B^2}{\left[A + K_k(1 + r_k)\right]^2 + B^2} \leq 0
\end{aligned}$$

which is a contradiction.

Notice that in the case when $\bar{r}_k = \bar{h} = 0$ for all k, $\lambda = 0$ is a solution of (12), so no definite conclusion can be drawn concerning the stability of the equilibrium.

4. THE LINEAR CASE

We assume here that the firms produce durable goods with life time of Y years, so at each year $100/Y\,\%$ of the goods in use by the consumers have to be replaced by new items. Introduce the notation $\gamma = 100/Y$ and $\alpha = 1 - \gamma$, and let Q denote the amount of goods being with the consumers. Then at the end of each year the total volume of goods being possessed by the consumers is $\sum_{k=1}^{n} x_k + \alpha Q$, and the change in Q equals $\sum_{k=1}^{n} x_k - \alpha Q$, since the consumers buy the amount $\sum_{k=1}^{n} x_k$ but they have to replace αQ.

For the sake of simplicity assume that the price and cost functions are linear:

$$f\left(\sum_{k=1}^{n} x_k, Q\right) = B - A\left(\sum_{k=1}^{n} x_k + \alpha Q\right), \qquad C_k(x_k) = c_k + d_k x_k,$$

and equation (1) has the special form

$$\dot{Q} = \sum_{k=1}^{n} x_k - \gamma Q, \tag{13}$$

where all parameters are positive. In this special case, $h = 1$, $\bar{h} = -\gamma$, and the best reply of firm k is obtained by maximizing function

$$x_k \left(B - A(x_k + S_k + \alpha Q)\right) - \left(c_k + d_k x_k\right). \tag{14}$$

Assuming interior optimum, the first order conditions imply that

$$B - 2Ax_k - AS_k - A\alpha Q - d_k = 0,$$

so

$$x_k = R_k(S_k, Q) = -\frac{1}{2} S_k - \frac{\alpha}{2} Q + \frac{B - d_k}{2A}. \tag{15}$$

Therefore,

$$r_k = \frac{\partial R_k}{\partial S_k} = -\frac{1}{2} \qquad \text{and} \qquad \bar{r}_k = \frac{\partial R_k}{\partial Q} = -\frac{\alpha}{2}.$$

Clearly, Conditions (A), (B), (C), and the additional assumption of the theorem are all satisfied, so the equilibrium is locally asymptotically stable. System (4) is linear, where local asymptotic stability of the equilibrium implies global asymptotical stability, so under the conditions of the Theorem the equilibrium is globally asymptotically stable.

5. CONCLUSIONS

In this paper, the local asymptotic stability of the equilibrium in single product oligopolies was examined without product differentiation and with intertemporal demand interaction. We have derived conditions for the local asymptotical stability of the equilibrium. Single product oligopoly was selected for the sake of mathematical simplicity. Models with product differentiation and multi-product models can be examined in a similar way.

Condition (A) and the first part of Condition (B) are common assumptions in oligopoly theory, and imply the strict concavity of the profit functions. This concavity plays an important role in the uniqueness of the best replies and also in the existence of a unique pure Nash equilibrium without intertemporal demand interaction[3]. The second part of Condition (B) shows that the dependence of the price function on Q is similar to its dependence on S; however, we do not assume any relation between f'_Q and C''_k as we have done in Condition (A) about f'_S. Condition (C) requires that the rate of increase in Q is an increasing function of the total production of industry and it decreases in Q. This assumption certainly holds in saturated markets, since more product adds more to market saturation, and more earlier sales result in more later replacements. We have also shown that in the linear case, all conditions hold necessarily, so the equilibrium is always globally asymptotically stable.

REFERENCES

1. A. Cournot, *Recherches sur les Principles Mathèmatiques de la Thèorie de Richesses*, Hachette, Paris, (1838); English translation (1960): Researches into the Mathematical Principles of the Theory of Wealth. Kelley, New York.
2. K. Okuguchi, *Expectations and Stability in Oligopoly Models*, Springer-Verlag, Berlin, (1976).
3. K. Okuguchi and F. Szidarovszky, *The Theory of Oligopoly with Multi-Product Firms*, (Second Edition), Springer-Verlag, Berlin, (1999).
4. K. Froot and P. Klemperer, Exchange rate pass-through when market share matters, *American Economic Review* **79**, 637–653, (1989).
5. T. Tivig, Exchange rate pass-though in two-period duopoly, *International Journal of Industrial Organization* **14**, 631–645, (1996).
6. K. Okuguchi and F. Szidarovszky, Oligopoly with intertemporal demand interaction, *Journal of Economic Research* **8**, 51–61, (2003).